International Journal of Engineering & Scientific Research
Vol. 6 Issue 2, February 2018,
ISSN: 2347-6532 Impact Factor: 6.660
Journal Homepage: http://www.esrjournal.com Email: esrjeditor@gmail.com Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

On b*ĝ - homeomorphism in Topological Spaces

K.Bala Deepa Arasi [1], Assistant Professor of Mathematics, A.P.C. Mahalaxmi College for Women, Thoothukudi, TN, India.

M.Mari Vidhya [2], M.Phil Scholar, St. Mary's College (Autonomous), Thoothukudi, TN, India.

Abstract: In this paper, we define a new class of function namely b*ĝ-homeomorphisms and we prove some of their basic properties. Also, we prove some of their relationship among other homeomorphisms. Throughout this paper $f:(X,\tau) \to (Y,\sigma)$ is a function from a topological space (X,τ) to a topological space (Y,σ).

Keywords: b*ĝ-closed sets, b*ĝ-continuous functions, b*ĝ-open maps, b*ĝ-closed maps and b*ĝ-homeomorphisms.

AMS Mathematics Subject Classification (2010): **54A05**.

1. Introduction:

In 1996, D. Andrijevic [1] introduced b-open sets in topology and studied its properties. In 1970, N.Levine [9] introduced generalized closed sets and studied their basic properties. In 2003, M.K.R.S.Veerakumar [15] defined ĝ-closed sets in topological spaces and studied their properties. b*-closed sets have been introduced and investigated by Muthuvel [11] in 2012. In 2016, K.Bala Deepa Arasi and G.Subasini [2] introduced b*ĝ-closed sets and studied its properties. K.Balachandran et al introduced the concept of generalized continuous maps in Topological spaces. In 2017, we [3] introduced b*ĝ-continuous functions and b*ĝ-open maps in topological spaces and also studied their properties.

Now, we define a new version of map called b*ĝ-homeomorphism. Also, we prove some of its properties and establish the relationships between b*ĝ-homeomorphisms and other known existing homeomorphisms.

2. Preliminaries:

Throughout this paper (X,τ) (or simply X) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X,τ), Cl(A), Int(A) and A^c denote the closure of A, interior of A and the complement of A respectively. We are giving some basic definitions.

Definition: 2.1 A subset A of a topological space (X,τ) is called

1. a semi-open set[10] if A⊆Cl(Int (A)).
2. an α-open set[4] if A⊆Int(Cl(Int(A))).
3. a b-open set [1] if A⊆Cl(Int(A)) ∪ Int(Cl(A)).
4. a regular open set[12] if A=Int(Cl(A)).

The complement of a semi-open (resp. α-open, regular open) set is called semi-closed (resp. α-closed, regular closed) set. The intersection of all semi-closed (resp. α-closed, regular closed) sets of X containing A is called the semi-closure (resp. α-closure, regular closure) of A and is denoted by sCl(A) (resp. αCl(A), rCl(A)). The family of all b*ĝ-open (resp. α-open, semi-open, b-open, regular open) subsets of a space X is denoted by b*ĝO(X) (resp. αO(X), sO(X), bO(X), rO(X)).

Definition: 2.2 A function $f:(X,\tau)\to(Y,\sigma)$ is called a

1. continuous[15] if $f^{-1}(V)$ is closed in X for every closed set V in Y.
2. semi-continuous[7] if $f^{-1}(V)$ is semi-closed in X for every closed set V in Y.
3. α-continuous[4] if $f^{-1}(V)$ is α-closed in X for every closed set V in Y.
4. regular continuous[12] if $f^{-1}(V)$ is regular closed in X for every closed set V in Y.
5. gs-continuous[5] if $f^{-1}(V)$ is gs-closed in X for every closed set V in Y.
6. gb-continuous[16] if $f^{-1}(V)$ is gb-closed in X for every closed set V in Y.
7. bĝ-continuous[14] if $f^{-1}(V)$ is bĝ -closed in X for every closed set V in Y.
8. g*s-continuous[13] if $f^{-1}(V)$ is g*s-closed in X for every closed set V in Y.
9. gr*-continuous[8] if $f^{-1}(V)$ is gr*-closed in X for every closed set V in Y.
10. (gs)*-continuous[6] if $f^{-1}(V)$ is (gs)*-closed in X for every closed set V in Y.
11. b*ĝ- continuous[3] if $f^{-1}(V)$ is b*ĝ-closed in X for every closed set V in Y.

Definition: 2.3 A function $f:(X,\tau)\to(Y,\sigma)$ is called a

1. open map[15] if $f(V)$ is open in Y for every open set V in X.
2. semi-open map[7] if $f(V)$ is semi-open in Y for every open set V in X.
3. α-open map[4] if $f(V)$ is α-open in Y for every open set V in X.
4. regular open map[12] if $f(V)$ is regular open in Y for every open set V in X.
5. gs-open map[5] if $f(V)$ is gs-open in Y for every open set V in X.
6. gb-open map[16] if $f(V)$ is gb-open in Y for every open set V in X.
7. bĝ-open map[14] if $f(V)$ is bĝ-open in Y for every open set V in X.
8. g*s-open map[13] if $f(V)$ is g*s-open in Y for every open set V in X.
9. gr*-open map[8] if $f(V)$ is gr*-open in Y for every open set V in X.
10. (gs)*-open map[6] if $f(V)$ is (gs)*-open in Y for every open set V in X.
11. b*ĝ-open map[3] if $f(V)$ is b*ĝ-open in Y for every open set V in X.

Remark:2.4 The family of all b*ĝ-closed (resp.α-closed, semi-closed, b-closed, regular closed) subsets of a space X is denoted by b*ĝC(X) (resp. αC(X), sC(X), bC(X), rC(X)).

3. b*ĝ-Homeomorphism:

We introduce the following definition.

Definition 3.1: A bijection $f:(X,\tau)\to(Y,\sigma)$ is called a **b*ĝ-homeomorphism** if f is both b*ĝ-continuous map and b*ĝ-open map.
That is, both f and f^{-1} are b*ĝ-continuous map.

Example 3.2: Let $X = Y$ = {a,b,c} with topologies τ = {X, ϕ,{a},{a,b}} and σ ={Y,ϕ, {a},{b},{a,b}}. Define a function $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=b$, $f(b)=a$, $f(c)=c$. b*ĝC(X)={X,ϕ,{b},{c},{a,c},{b,c}} and C(Y)={Y,ϕ,{c}, {a,c},{b,c}} and b*ĝO(Y)={Y,ϕ, {a},{b},{a,b}, {b,c},{a,c}}. Here the inverse image of C(Y) {c},{b,c} and {a,c} are {c},{a,c} and {b,c} which are b*ĝC(X) and the image of O(X) {a} and {a,b} are {b} and {a,b} which are b*ĝO(Y) . Hence f is b*ĝ-homeomorphism.

Proposition 3.3:

a) Every homeomorphism is b*ĝ-homeomorphism
b) Every α-homeomorphism is b*ĝ-homeomorphism
c) Every semi-homeomorphism is b*ĝ-homeomorphism
d) Every regular homeomorphism is b*ĝ-homeomorphism
e) Every gr*-homeomorphism is b*ĝ-homeomorphism
f) Every (gs)*-homeomorphism is b*ĝ-homeomorphism
g) Every g*s-homeomorphism is b*ĝ-homeomorphism

Proof:

a) Let $f:(X,\tau)\to(Y,\sigma)$ be a homeomorphism. Then f is continuous and open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.
b) Let $f:(X,\tau)\to(Y,\sigma)$ be a α-homeomorphism. Then f is α-continuous and α-open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.
c) Let $f:(X,\tau)\to(Y,\sigma)$ be a semi-homeomorphism. Then f is semi-continuous and semi-open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.
d) Let $f:(X,\tau)\to(Y,\sigma)$ be a regular homeomorphism. Then f is regular continuous and regular open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.
e) Let $f:(X,\tau)\to(Y,\sigma)$ be a gr*-homeomorphism. Then f is gr*-continuous and gr*-open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.
f) Let $f:(X,\tau)\to(Y,\sigma)$ be a (gs)*-homeomorphism. Then f is (gs)*-continuous and (gs)*-open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.
g) Let $f:(X,\tau)\to(Y,\sigma)$ be a g*s-homeomorphism. Then f is g*s-continuous and g*s-open map. By proposition 3.5 and 4.4 in [3], f is b*ĝ-continuous and b*ĝ-open map. Hence f is b*ĝ-homeomorphism.

The following examples show that the converse of the above proposition need not be true.

Example 3.4:

a) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{a,c}} and σ = {Y,ϕ,{b },{c},{b,c}}. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=b$, $f(b)=c$, $f(c)=a$. b*ĝC(X) = {X,ϕ,{b},{a,b},{b,c}}; b*ĝO(Y) = {Y,ϕ,{b},{c},{a,b},{b,c},{a,c}}; C(Y) = {Y,ϕ,{a},{a,b},{a,c}} and C(X) = {X,ϕ,{b}}. Here, the inverse image of C(Y) {a},{a,b} and {a,c} are {b},{b,c} and {a,b} which are b*ĝC(X) but not C(X). Hence f is b*ĝ-homeomorphism but not a homeomorphism, since f is not a continuous map.

b) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{a},{c},{a,c},{b,c}} and σ = {Y,ϕ,{b},{c},{b,c}}. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=b, f(b)=a, f(c)=c$. b*ĝC(X) = {X, ϕ,{a},{b},{a,b},{b,c}}; b*ĝO(Y) = {Y,ϕ,{b},{c},{a,b},{b,c},{a,c}}; C(Y) = {Y,ϕ,{a},{a,b},{a,c}}; αO(Y) = {Y,ϕ,{b},{c},{b, c}} and αC(X)={ X,ϕ,{a},{a,b},{b,c}}. Here, the image of O(X) {a},{c},{a,c} and {b,c} are {b}, {c},{b,c} and {a,c} which are b*ĝO(Y) but not αO(Y). Hence f is b*ĝ-homeomorphism but not a α-homeomorphism, since f is not a α-open map.

c) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{c},{a,b}} and σ = {Y,ϕ,{a}}. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=c, f(b)=a$, $f(c)=b$. b*ĝC(X) = {X,ϕ,{c},{a,b}}; b*ĝO(Y) = {Y,ϕ,{a},{b},{c}, {a,b},{a,c}}; C(Y) = {Y,ϕ,{b,c}}; sO(Y) = {Y,ϕ,{a},{a,b},{a,c}} and sC(X) = {X,ϕ,{c},{a,b}}. Here, the image of O(X) {c} and {a,b} are {b} and {a,c} which are b*ĝO(Y) but not sO(Y). Hence f is b*ĝ-homeomorphism but not a semi-homeomorphism, since f is not a semi-open map.

d) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{a},{a,b}} and σ = {Y,ϕ,{b} }. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=b, f(b)=a$, $f(c)=c$. b*ĝC(X) = {X,ϕ,{b},{c},{a,c},{b,c}}; b*ĝO(Y) = {Y,ϕ,{a}, {b},{c},{a,b},{b,c}}; C(Y) = {Y,ϕ,{a,c}}; rO(Y) = {Y,ϕ} and rC(X) = {X,ϕ}. Here, the image of O(X) {a} and {a,b} are {b} and {a,b} which are b*ĝO(Y) but not rO(Y). Hence f is b*ĝ-homeomorphism but not a regular homeomorphism, since f is not a regular open map.

e) Let X=Y={a,b,c} with topologies τ ={X, ϕ,{a},{b},{a,b}} and σ ={Y,ϕ, {a}}. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=b$, $f(b)=a, f(c)=c$. b*ĝC(X) = {X,ϕ,{a},{b},{c},{a,c},{b,c}}; b*ĝO(Y) = {Y,ϕ, {a},{b},{c},{a,b},{a,c}}; C(Y) = {Y,ϕ,{b,c}}; gr*O(Y) = {Y,ϕ,{a}} and gr*C(X) = {X,ϕ,{c},{a,c}, {b,c}}. Here, the image of O(X) {a},{b} and {a,b} are {b},{a} and {a,b} which are b*ĝO(Y) but not gr*O(Y). Hence f is b*ĝ-homeomorphism but not gr*-homeomorphism, since f is not gr*-open map.

f) Let X=Y={a,b,c} with topologies τ ={X, ϕ,{a},{b},{a,b}} and σ ={Y,ϕ, {a}}. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=a$, $f(b)=c, f(c)=b$. b*ĝC(X) = {X,ϕ,{a},{b},{c},{a,c},{b,c}}; b*ĝO(Y) = {Y,ϕ, {a},{b},{c},{a,b},{a,c}}; C(Y) = {Y,ϕ,{b,c}}; (gs)*O(Y) = {Y,ϕ,{a}} and (gs)*C(X) = {X,ϕ,{c},{a,c },{b,c}}. Here, the image of O(X) {a},{b} and {a,b} are {a},{c} and {a,c} which are b*ĝO(Y) but not (gs)*O(Y). Hence f is b*ĝ-homeomorphism but not (gs)*-homeomorphism, since f is not (gs)*-open map.

g) Let X=Y={a,b,c} with topologies τ ={X, ϕ,{a},{a,b}} and σ ={Y,ϕ,{b}}. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=c$, $f(b)=b, f(c)=a$. b*ĝC(X) = {X,ϕ,{b},{c},{a,c},{b,c}}; b*ĝO(Y) = {Y,ϕ,{a},{b},{c}, {a,b},{b,c}}; C(Y) = {Y,ϕ,{a,c}}; g*sO(Y) = {Y,ϕ,{b},{a,b},{b,c}} and g*sC(X) = {X,ϕ,{b},{c},{b,c }}. Here, the inverse image of C(Y) {a,c} is {a,c} which is b*ĝC(X) but not g*sC(X). Hence f is b*ĝ-homeomorphism but not g*s-homeomorphism, since f is not g*s-continuous map.

Proposition 3.5:

a) Every b*ĝ-homeomorphism is gs-homeomorphism

b) Every b*ĝ-homeomorphism is gb-homeomorphism

c) Every b*ĝ-homeomorphism is bĝ-homeomorphism

Proof:

a) Let $f:(X,\tau)\to(Y,\sigma)$ be a b*ĝ-homeomorphism. Then f is b*ĝ-continuous and b*ĝ-open map. By proposition 3.7 and 4.6 in [3], f is gs-continuous and gs-open map. Hence f is gs-homeomorphism.

b) Let $f:(X,\tau)\to(Y,\sigma)$ be a b*ĝ-homeomorphism. Then f is b*ĝ-continuous and b*ĝ-open map. By proposition 3.7 and 4.6 in [3], f is gb-continuous and gb-open map. Hence f is gb-homeomorphism.

c) Let $f:(X,\tau)\to(Y,\sigma)$ be a b*ĝ-homeomorphism. Then f is b*ĝ-continuous and b*ĝ-open map. By proposition 3.7 and 4.6 in [3], f is bĝ-continuous and bĝ-open map. Hence f is bĝ-homeomorphism.

The following examples show that the converse of the above proposition need not be true.

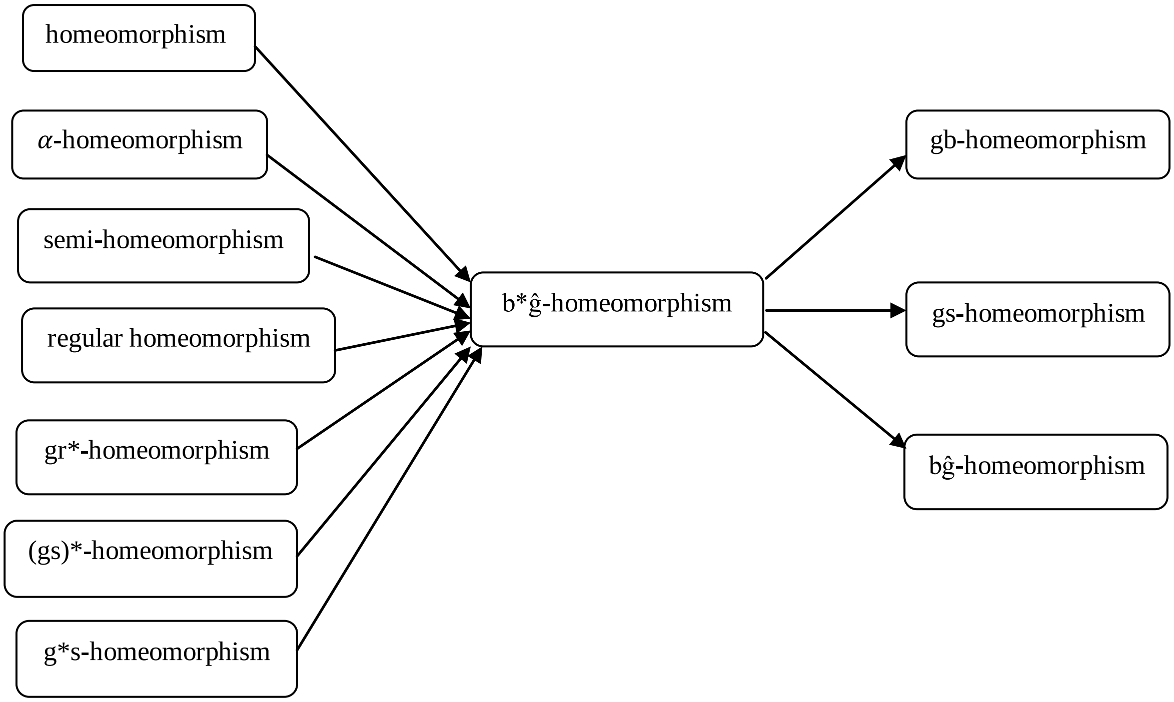
Example 3.6:

a) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{a}} and σ = {Y,ϕ,{a},{b,c}}. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=c$, $f(b)=b$, $f(c)=a$. b*ĝC(X) = {X,ϕ,{b},{c},{a,b},{b,c},{a,c}}; b*ĝO(Y) = {Y,ϕ,{a}, {b,c}}; C(Y) = {Y,ϕ,{a},{b,c}}; gsO(Y) = {Y,ϕ,{a},{b},{c},{a,b},{b,c},{a,c}} and gsC(X) = {X,ϕ,{b},{c},{a,b},{b,c},{a,c}}. Here, the image of O(X) {a} is {c} which is gsO(Y) but not b*ĝO(Y). Hence f is gs-homeomorphism but not a b*ĝ-homeomorphism, since f is not a b*ĝ-open map.

b) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{a},{b},{a,b},{a,c}} and σ ={Y,ϕ,{c}, {a,b}}. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=c$, $f(b)=a$, $f(c)=b$. b*ĝC(X) = {X,ϕ,{b},{c},{b,c},{a,c}}; b*ĝO(Y) = {Y,ϕ,{c},{a,b}}; C(Y) = {Y,ϕ,{c},{a,b}}; gbO(Y) = {Y,ϕ,{a},{b},{c},{a,b},{b,c},{a,c}} and gbC(X) = {X,ϕ,{b},{c},{b,c},{a,c}}. Here, the image of O(X) {a},{b},{a,b} and {a,c} are {c},{a}, {a,c} and {b,c} which are gbO(Y) but not b*ĝO(Y). Hence f is gb-homeomorphism but not a b*ĝ-homeomorphism, since f is not a b*ĝ-open map.

c) Let X = Y = {a,b,c} with topologies τ = {X, ϕ,{a},{a,b},{a,c}} and σ = {Y,ϕ,{a,b}}. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=a$, $f(b)=c$, $f(c)=b$. b*ĝC(X) = {X,ϕ,{b},{c},{b,c}}; b*ĝO(Y) = {Y,ϕ,{a},{b}, {a,b}}; C(Y) = {Y,ϕ,{c}}; bĝO(Y) = {Y,ϕ,{a},{b},{a,b},{b,c},{a,c}} and bĝC(X) = {X,ϕ,{b},{c},{b,c} }. Here, the image of O(X) {a},{a,b} and {a,c} are {a},{a,c} and {a,b} which are bĝO(Y) but not b*ĝO(Y). Hence f is bĝ-homeomorphism but not a b*ĝ-homeomorphism, since f is not a b*ĝ-open map.

Remark 3.7: The following diagram shows the relationships of b*ĝ-homeomorphism with other known existing homeomorphisms. A → B represents A implies B but not conversely.



Proposition 3.8: For any bijection $f:(X,\tau) \to (Y,\sigma)$ the following statements are equivalent

a) Its inverse map $f^{-1}: Y \to X$ is b*ĝ-continuous

b) f is b*ĝ-open map

c) f is b*ĝ-closed map

Proof: (a)⇒(b): Let G be any open set in X. Since f^{-1} is b*ĝ-continuous, the inverse image of G under f^{-1}, namely (G) is b*ĝ-open in Y and so, f is b*ĝ-open map.

(b)⇒(c): Let F be any closed set in X. Then F^c is open in X. Since f is b*ĝ-open, $f(F^c)$ is b*ĝ-open in Y. But $f(F^c)$ $=Y–(F)$ and so, $f(F)$ is b*ĝ-closed in Y. Therefore f is b*ĝ-closed map.

(c)⇒(a): Let F be any closed set in X. Then the inverse image of F under f^{-1}, namely $f(F)$ is b*ĝ-closed in Y, since f is b*ĝ-closed map. Therefore f^{-1} is b*ĝ-continuous.

Proposition 3.9: Let $f:(X,\tau) \to (Y,\sigma)$ be a bijective and b*ĝ-continuous map. Then the following statements are equivalent

a) f is b*ĝ-open map

b) f is b*ĝ-homeomorphism

c) f is b*ĝ-closed map

Proof: (a)⇒(b): Given $f:(X,\tau) \to (Y,\sigma)$ is a bijective, b*ĝ-continuous and b*ĝ-open. Then by definition 3.1, f is b*ĝ-homeomorphism.

(b)⇒(c): Given f is b*ĝ-open and bijective. By proposition 3.8, f is b*ĝ-closed map.

(c)⇒(a): Given f is b*ĝ-closed and bijective. By proposition 3.8, f is b*ĝ-open map.

4. References:

1) D. Andrijevic, On b-open sets, *Mat. Vesnik.,* 48(1996), no. 1-2, 59-64.

2) K.Bala Deepa Arasi and G.Subasini, On b*ĝ-closed sets in topological spaces, *International Research Journal of Mathematics, Engineering and IT*, Vol.2, Issue 12, December 2015.

3) K.Bala Deepa Arasi and M.Mari Vidhya, On b*ĝ-continuous functions and b*ĝ-open maps in topological spaces, *International Journal of Mathematical Archieve – 8(7), 2017, 121-128.*

4) M.Caldas and E.Ekici, On Slightly γ continuous functions *Bol. Soc, Parana Mat (3)22(2004) No.*2,63-74.

5) R.Devi, H.Maki and K.Balachandran, On Semi-generalized homeomorphisms and generalized semi-homeomorphism in topological spaces, *Indian J. Pure. Appl. Math.,* 26(3) (1995), 271-284.

6) L.Elvina Mary and R.Myvizhi, On (gs)*-closed sets in topological spaces, *International Journal of Mathematics Trends and Technology,* Vol 7-No. 2, March 2014.

7) J.Ferrer, V.Gregori and I.L.Reilly, Some properties of semi-continuous functions and Quasi-uniform spaces, *Mem.Fac.Sci.Kochi Univ.Ser.A. Math*, 15(1994), 57-63.

8) K.Indirani, P.Sathishmohan and V.Rajendran, On gr*-continuous functions in topological spaces, *IJSETR, Vol 3,* Issue 4, April 2014.

9) N.Levine, Generalized closed sets in topology *Rend. Circ. Mat. Palermo,*19 (1970) 89-96.

10) N.Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* 70(1963), 36-41.

11) Muthuvel and Parimelazhagan, On b*-closed sets in topological spaces, *Int. Journal of Math. Analysis*, Vol.6, 2012. No.47, 2317-2323.

12) A.Narmadha and Nagaveni, On regular b-closed sets in topological spaces, *Heber International Conference on Application of Mathematics and Statistics,* HICAMS-2012, 5-7, Jan 2012, 81-87.

13) A.Pushpalatha and K.Anitha, On g*s-closed sets in topological spaces, *International J. Contemp. Math. Sciences,* Vol 6, 2011, No. 19, 917-929.

14) R.Subasree and M.Maria Singam, On bĝ-Continuous maps and bĝ-Open maps in topological spaces, *IJERT*, Vol 2(10), October 2013.

15) M.K.R.S.Veerakumar, On ĝ-closed sets in topological spaces, *Bull. Allahabad. Math. Soc.,* Vol.18, (2003), 99-112.

16) D.Vidhya and R.Parimelazhagan, On g*b-homeomorphism and contra g*b-continuous maps in topological spaces, *International Journal of Computer Applications,* Vol 58-No.14, November 2012.